

Spencer Manifolds

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Abstract

Almost-complex and hyper-complex manifolds are considered in this paper from the point of view of complex analysis and potential theory. The idea of holomorphic coordinates on an almost-complex manifold (M, \mathbf{J}) is suggested by D. Spencer [Sp]. For hypercomplex manifolds we introduce the notion of hyper- holomorphic function and develop some analogous statements. Elliptic equations are developed in a different way than D. Spencer . In general here we describe only the formal aspect of the developed theory.

0.1 INTRODUCTION.

Differentiable manifolds are described locally by smooth real coordinates. This is typical in differential geometry. Complex-analytic manifolds are equipped locally by complex-analytic coordinates. This give rise to the possibility of applying the theory of holomorphic functions of many complex variables in the local geometry of complex-analytic manifolds. In the case of almost complex manifolds (M, \mathbf{J}) one use ordinary real coordinates (x^1, \dots, x^{2n}) . Here we shall consider complex self-conjugate coordinates $(z^1, \dots, z^n, \bar{z}^1, \dots, \bar{z}^n)$, where $z^k = x^{2k-1} + ix^{2k}$ and $\bar{z}^k = x^{2k-1} - ix^{2k}$.

We denote by \mathbf{J}^* the action of \mathbf{J} on differential forms of M , i. e. by definition $(\mathbf{J}^*(\omega)X \stackrel{def}{=} \omega(\mathbf{J}X))$, where X is a vector field, and ω is a differential form on M .

For a fixed index k , we say that z^k is a "holomorphic" coordinate if $\mathbf{J}^*dz^k = idz^k$ and $\mathbf{J}^*d\bar{z}^k = -id\bar{z}^k$. For non-holomorphic coordinates z^q we have

$$\mathbf{J}^*dz^q = J_q^1 dz^1 + \dots + J_q^n dz^n + J_q^{n+1} d\bar{z}^{n+1} + \dots + J_q^{2n} d\bar{z}^{2n}$$

In the case z^k is a holomorphic coordinate for each $k = 1, \dots, n$, the almost complex structure \mathbf{J} is an integrable one. The interest of the existence of holomorphic coordinates z^k when the index k takes not all values $1, \dots, n$ is suggested by Donald Spencer [Sp].

By $\mathbb{H} = \mathbb{H}(\mathbf{1}, i, j, k), ij=k$, we will denote the 4-dimensional quaternionic vector space, i. e. $q \in \mathbb{H}$ means that $q = x^0 + ix^1 + jx^2 + kx^3$, where $x^0, x^1, x^2, x^3 \in \mathbb{R}$. We will use different complex number representation for quaternions q , namely $q = z + \zeta j$, where $z = x^0 + ix^1$ and $\zeta = x^2 + ix^3$. So we obtain the right j -complex splitting of \mathbb{H} , denoted by \mathbb{H}^j , i. e. $\mathbb{H}^j = (\mathbb{R} \oplus i\mathbb{R}) \oplus (\mathbb{R} \oplus i\mathbb{R})j$. By $\mathbb{R} \oplus i\mathbb{R}$ is denoted the tensor product of \mathbb{R} with itself under the basis $(1, 0)$ and $(0, i)$. Identifying $\mathbb{R} \oplus i\mathbb{R}$ with \mathbb{C} we have that \mathbb{H}^j is homeomorphic to $\mathbb{C} \times \mathbb{C}$. Analogously, we will consider the right i -complex splitting of \mathbb{H} , namely $\mathbb{H}^i = (\mathbb{R} \oplus j\mathbb{R}) \oplus (\mathbb{R} \oplus j\mathbb{R})i$, i.e. $q = x^0 + jx^3 + (x^2 - jx^4)i$. \mathbb{H}^i is homeomorphic to $\mathbb{C} \times \mathbb{C}$ too.

By \mathbb{H}^n is denoted the n -dimensional quaternionic vector space (real $4n$ -dimensional)

$$\mathbb{H}^n = \{(q^1, \dots, q^n) : q^\alpha \in \mathbb{H}, \alpha = 1, \dots, n\}$$

According to the above accepted notation we have $q^\alpha = z^\alpha + \zeta^\alpha j$, $\alpha = 1, \dots, n$ or

$$\mathbb{H}^n = \mathbb{C}^n + \mathbb{C}^n j, \quad \mathbb{C}^n = \{z^1, \dots, z^n : z^\alpha \in \mathbb{C}\}$$

This representation is with respect of the right j -complex splitting \mathbb{H}^j . A similar representation of \mathbb{H}^n can be written with respect to the right i -complex splitting $\mathbb{H}^i : \mathbb{H}^i = \mathbb{C}^n + \mathbb{C}^n i$, $\mathbb{C}^n = \mathbb{R}^n \oplus \mathbb{R}^n i$, $\mathbb{C}^n = \mathbb{R}^n \ominus \mathbb{R}^n i$.

Let $(M, \mathbf{J}, \mathbf{K})$ be a hyper-complex manifold, $\mathbf{J}\mathbf{K} + \mathbf{K}\mathbf{J} = 0$, $\dim M_{\mathbb{R}} = 4n$. A pair of complex coordinates (z, ζ) is called *hyper-holomorphic pair* if z is holomorphic with respect to the almost-complex manifold (M, \mathbf{J}) and ζ is holomorphic with respect to (M, \mathbf{K}) .

0.2 Holomorphic coordinates

0.2.1 Almost-holomorphic functions

By definition a function $f : U \rightarrow \mathbf{C}$ where U is an open subset of M , is called *almost holomorphic* or almost complex if $\bar{\partial}f = 0$. The above definition can be reformulated in the following equivalent form:

$$f \text{ is almost holomorphic iff } \mathbf{J}^*df = idf$$

Respectively, f is *almost-antiholomorphic* iff $\mathbf{J}^*df = -idf$. For the proof of the equivalence it is enough to take in view that the exterior derivative d is decomposed as $d = \partial + \bar{\partial}$ over the space of smooth functions on M . Another form of this definition is obtained taking the real and imaginary parts of f , i.e. $f = u + iv$. In view of $df = du + idv$ we receive $\mathbf{J}^*du + i\mathbf{J}^*dv = idu - dv$. This means that $\mathbf{J}^*du = -dv$ and $\mathbf{J}^*dv = du$. As the obtained two equations are

not independent, we can state the following Cauchy- Riemann type form of the definition

$f = u + iv$ is almost-holomorphic iff $\mathbf{J}^* dv = du$ or equivalently $\mathbf{J}^* du = -dv$.

Respectively: $f = u + iv$ is almost-anti holomorphic iff $\mathbf{J}^* dv = -du$ or equivalently $\mathbf{J}^* du = dv$

Remark: For an almost complex manifold (M, \mathbf{J}) with non-integrable \mathbf{J} , the decomposition $d = \partial + \bar{\partial}$ is not valid over differential (p, q) -forms on (M, \mathbf{J}) .

The following proposition is well-known:

Proposition 1. The almost complex structure \mathbf{J} of the almost complex manifold (M, \mathbf{J}) , $\dim_{\mathbf{R}} M = 2n$, is an integrable almost complex structure if and only if for every point $p \in M$, there is a neighborhood U of p and almost holomorphic functions $f_j : U \rightarrow \mathbb{C}$, $j = 1, \dots, n$, which differentials at p , i.e. $d_p f_j$, $j = 1, \dots, n$, are \mathbb{C} -linear independent.

Remark: Taking $(U; f_1, \dots, f_n)$ as local coordinate system (as f_j are functionally independent on a neighborhood of p), we obtain a local complex-analytic coordinate system $(U; z_1, \dots, z_n)$, where $z^k = f_k$.

0.2.2 Spencer coordinates

We say that a local Spencer coordinate system of type m is defined on an almost complex manifold (M, \mathbf{J}) if the following two conditions hold:

- 1.) There exist an open subset U of M and m different functionally independent almost holomorphic functions $f_j : U \rightarrow \mathbb{C}$, $j = 1, \dots, m$, such that
- 2.) The sequence (f_1, \dots, f_m) is a maximal sequence of functionally independent on U almost-holomorphic functions.
- 3.) The sequence

$$(U, w^1, \dots, w^m, z^{m+1}, \dots, z^n, \bar{w}^{n+1}, \dots, \bar{w}^{n+m}, \bar{z}^{n+m+1}, \dots, \bar{z}^{2n})$$

where $w^j = f_j$, $j = 1, \dots, m$, determines a local self-conjugate system on (M, \mathbf{J}) .

An almost complex manifold which is equipped with an atlas of local Spencer coordinate systems is by definition an almost-complex manifold of Spencer type m . It is to remark that the notion of Spencer type is correctly defined in the category of almost complex manifolds. This follows by the fact that each composition of almost-holomorphic mappings and each inverse of almost-holomorphic diffeomorphism are almost-holomorphic too.

Lemma 1: The matrix representation of \mathbf{J}^* in each local Spencer coordinate system

$$(U, w^1, \dots, w^m, z^{m+1}, \dots, z^n, \bar{w}^{n+1}, \dots, \bar{w}^{n+m}, \bar{z}^{n+m+1}, \dots, \bar{z}^{2n})$$

where $w^j = f_j$, $j = 1, \dots, m$, are functionally independent almost holomorphic functions, seems as follows

$$\begin{pmatrix} iE_m & * & 0 & * \\ 0 & * & 0 & * \\ 0 & * & -iE_m & * \\ 0 & * & 0 & * \end{pmatrix}$$

E_m being the unit $m \times m$ matrix.

Proof. It is enough to take in view that:

$$(dw^1, \dots, dw^m, dz^{m+1}, \dots, dz^n, d\bar{w}^{n+1}, \dots, d\bar{w}^{n+m}, dz^{n+m+1}, \dots, dz^{2n})$$

is basis of the cotangent space and

$$\mathbf{J}^* dw^j = \mathbf{J}^* df_j = idf_j = idw^j, \quad j = 1, \dots, m \quad \blacksquare$$

Consequences: The first m equations of the system $J^* df = idf$ are just the conditions $\partial f / \partial \bar{z}_j = 0, j = 1, \dots, m$

We shall consider the mapping from U to \mathbb{C}^m defined by f_1, \dots, f_m . This mapping is a smooth submersion as it can be considered as a composition of the diffeomorphism defined by Spencer coordinates of U in $\mathbb{C}^n \times \bar{\mathbb{C}}^n$ and the projection of $\mathbb{C}^n \times \bar{\mathbb{C}}^n$ on $\mathbb{C}^m, m < n$. This mapping will be denoted by f_U , and the image of U by f_U will be denoted U_m^c . It is an open subset of \mathbb{C}^m , which will be called a naturally associated m -dimensional open set to the considered local Spencer coordinate system.

Lemma 2: Each almost holomorphic function h , defined on a local Spencer coordinate system U is represented as a superposition of a holomorphic function H defined on U_m^c and the almost holomorphic functions f_1, \dots, f_m defined on U , i.e.

$$h = H \circ (f_1, \dots, f_m) = H(f_1, \dots, f_m)$$

Proof: As $w_j = f_j, j = 1, \dots, m$, is a system of smooth functionally independent on U functions, we have $h = H(w^1, \dots, w^m)$ with $H \in C^\infty(U)$. But

$$\bar{\partial} H = (\bar{\partial} H / \partial \bar{w}^1) d\bar{w}^1 + \dots + (\partial H / \partial \bar{w}^m) d\bar{w}^m$$

and in view of $\bar{\partial} H = \bar{\partial} h = 0$, we get that the above written (0,1)-form is a zero-form, or $\partial H / \partial \bar{w}_j = 0, j = 1, \dots, m$. \blacksquare

Lemma 3: Let (w^1, \dots, w^m) and (v^1, \dots, v^m) be two systems of holomorphic coordinates on U_m^c defined by two different systems of almost holomorphic on U systems (f_1, \dots, f_m) and (h_1, \dots, h_m) . Then there exists a bijective holomorphic transition mapping between the mentioned two coordinate systems.

Proof. According to *Lemma 2* we have $v_j = H_j(w^1, \dots, w^m), j = 1, \dots, m$, where H_j are holomorphic functions of (w_1, \dots, w_m) . The system $H = (H_1, \dots, H_m)$ defines the mentioned transition mapping as the differentials dH_j which are \mathbb{C} -linear independent. \blacksquare

Recapitulating we obtain the following

Proposition 2: On each paracompact almost complex manifold (M, J) of constant Spencer type m there exists a locally finite covering U_j by self-conjugated Spencer's coordinate system $(U_j, z_j^1, \dots, z_j^m, \dots)$ such that in every intersection $U_j \cap U_k$ the holomorphic coordinates z_j^1, \dots, z_j^m change holomorphically in the other holomorphic coordinates z_k^1, \dots, z_k^m .

0.2.3 Local submersions and local foliations

As it was remarked above the mapping $f_U : U \rightarrow \mathbb{C}^m$, defined by the almost holomorphic functions (f_1, \dots, f_m) is a local submersion. According to introduced notations

$$f_U(U) = U_m^c \subset \mathbb{C}^m$$

The leaves of this submersion are defined as the stalks of the mapping f_U . Each leaf is a smooth $(2n - 2m)$ -dimensional submanifold of U on which all functions f_j have constant value. Transversal leaves are defined as univalent inverse images of U_m^c , i.e. as sections of U over U_m^c .

We shall consider the set of all open subsets $U_m^c \subset \mathbb{C}^m$, corresponding to different mappings f_U , U open subset of M . This set together with the transition mappings described in *Lemma 3* defines a pseudo-group of holomorphic transition mappings between open subsets of \mathbb{C}^m denoted as follows

$$\Gamma\{U_m^c, V_m^c, \dots; H : U_m^c \rightarrow V_m^c, \dots\}$$

We shall denote by \mathbb{C}^m/Γ the set of equivalent points of \mathbb{C}^m with respect to the natural equivalence defined by the holomorphic transition mappings. With this in mind we consider the family $\{f_U : U \rightarrow M\}$ and will define a glued mapping

$$f : M \rightarrow \mathbb{C}^m/\Gamma$$

as follows: if $p \in M$ we take an open subset U such that $p \in U$ and we set

$$f(p) = \{\text{the equivalence class of the point } f_U(p)\}$$

Under the assumption that \mathbb{C}^m/Γ is equipped with the standard complex structure \mathbf{i} defined by holomorphic coordinates (w^1, \dots, w^m) we can formulate the following

Lemma 4. The glued mapping $f : M \rightarrow \mathbb{C}^m/\Gamma$ is an almost holomorphic mapping between (M, \mathbf{J}) and $(\mathbb{C}^m/\Gamma, \mathbf{i})$.

Proof. As the glued mapping f coincides locally with some f_U we have:

$\mathbf{J}^* df_U = \mathbf{J}^* d(f_1, \dots, f_m) = \mathbf{J}^* (df_1, \dots, df_m) = (\mathbf{J}^* df_1, \dots, \mathbf{J}^* df_m) = \mathbf{i}(df_1, \dots, df_m) = id f_U$. So each f_U is an almost holomorphic mapping ■

Lemma 5. The sheaf of almost holomorphic functions on M is the inverse image of the sheaf of holomorphic functions on \mathbb{C}^m/Γ .

Proof. The mentioned sheaf on M is defined by the presheaf $\{U, \mathcal{O}_M(U)\}$ where U varies in the set of all open subsets of M and $\mathcal{O}_M(U)$ is defined as follows:

$$\mathcal{O}_M(U) = \{h \circ f_U \mid h \in \mathcal{O}_{\mathbb{C}^m/\Gamma}(f_U(U))\}.$$

0.2.4 Hypercomplex manifolds and hyperholomorphic functions

Let M be a $4n$ -dimensional (\mathcal{C}^∞) smooth manifold. A hypercomplex structure on M is defined by a pair of two almost complex structures \mathbf{J} and \mathbf{K} such

that $\mathbf{JK} + \mathbf{KJ} = 0$. It is easy to see that the composition \mathbf{JK} is an almost-complex structure too. Moreover, for each triple of real numbers b, c, d , such that $b^2 + c^2 + d^2 = 1$, the linear combination $b\mathbf{J} + c\mathbf{K} + d(\mathbf{JK})$ is an almost-complex structure on M . So there is a family of almost complex structures on M parametrized by the points of sphere Σ^2 . (See for instance [AM], [ABM]).

We shall consider almost-holomorphic functions on hypercomplex manifolds. The definition remain the same as in the above considered case, for instance on $(M, \mathbf{J}, \mathbf{K})$ we have \mathbf{J} -almost- holomorphic function which are complex-valued function f on (M, \mathbf{J}) such that $J^*df = idf$ using the right-side j -complex splitting of \mathbb{H} . Respectively \mathbf{K} -almost- holomorphic functions g on $(M, \mathbf{J}, \mathbf{K})$ are the almost-holomorphic with respect to (M, \mathbf{K}) such that $K^*dg = jdg$ using an i -complex splitting of \mathbb{H} .

Let $(M, \mathbf{J}, \mathbf{K})$ be a hypercomplex manifolds and \mathbb{H} be 4-dimensional quaternionic vector space. According to Sommese [So] the right-side multiplication by i and j are given respectively by the matrices S and T , called standard quaternionic structures.

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

In the paper of Sommese the matrix T is denoted by K .

As we have $S^2 = -\mathbf{1}, T^2 = -\mathbf{1}, (ST)^2 = -\mathbf{1}$, and $ST + TS = 0$, we can consider (\mathbb{H}, S, T) as a special hypercomplex manifold. (See [So]). A function F defined on an open subset $U \subset M$ with valued in \mathbb{H} is called \mathbf{J} -hyper-holomorphic function on U if $dF \circ J = S \circ dF$, or $J^*dF = SdF$. Using the right-side j -complex splitting \mathbb{H}^j we take the compositions of F with the projections of \mathbb{H} on the first and the second components of \mathbb{H}^j . So F is represented by a pair of complex valued functions denoted respectively by f and φ . If we set $F = u + i v + j \zeta + k \eta$, where u, v, ζ, η are real-valued functions on U , we can write $\varphi = u + iv + (\zeta + i\eta)j$, with $f = u + iv$, $\varphi = \zeta + i\eta$. Complexifying the matrix S , i.e. setting

$$S = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and taking $dF = df + d\varphi j$, we calculate that

$$J^*df + J^*d\varphi j = i df - i d\varphi j.$$

Having in mind the splitting \mathbb{H}^j , we get $J^*df = i df$ and $J^*d\varphi = -i d\varphi$, which means that f is \mathbf{J} -almost-holomorphic function on U and φ is \mathbf{J} -almost-antiholomorphic.

For the definition of \mathbf{K} -hyper-holomorphic function on U we shall use the other complex splitting of \mathbb{H} , namely \mathbb{H}^i . A function $G : M \rightarrow \mathbb{H}^i$, i.e. $G = g + \psi i$, $g = u' + j\zeta'$, $\psi = v' - j\eta'$, will be called \mathbf{K} -hyper-holomorphic function on U if $dG \circ K = T \circ dG$ or $K^*dG = TdG$. Taking a (2×2) -representation of the matrix T , i.e.

$$T = \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix}, \quad \mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

after a short calculation we get

$$K^*dg + i K^*d\psi = d\psi - i dg$$

It follows that $K^*dg = d\psi$ and $K^*d\psi = -dg$. This result is in terms of \mathbb{H}^i .

Now we will translate the obtained result in terms of \mathbb{H}^j . From $K^*(du' + d\zeta'j) = dv' - d\eta'j$ we get

$$K^*du' = dv' \quad \text{and} \quad K^*d\zeta' = -d\eta'.$$

Analogously, from $K^*(dv' - d\eta'j) = -(du' + d\zeta'j)$ we get

$$K^*dv' = -du' \quad \text{and} \quad K^*d\eta' = -d\zeta'.$$

But the system $K^*du' = dv'$, $K^*dv' = -du'$ is just the Cauchy-Riemann system, which says that the function $u' + iv'$ is **J**-almost-antiholomorphic, i. e. $J^*d(u' + iv') = -id(u' + iv')$. The function $\zeta' + i\eta'$ is **J**-almost-holomorphic.

0.2.5 Hyper-Spencer coordinates

Hyper-holomorphic coordinates on a hyper-complex manifold $(M, \mathbf{J}, \mathbf{K})$ can be introduced by functionally independent quaternionic-valued functions $f_\alpha + \varphi_\alpha j$, $\alpha = 1, \dots, m$, $m = (1/2)\dim M_{\mathbb{R}}$, or by the complex-valued function $(f_\alpha, \varphi_\alpha)$. We are interested of the possibility to have $m < (1/2)\dim M_{\mathbb{R}}$. More precisely, a **J**-hyper- Spencer coordinate system is defined locally on M as a maximal system of m functionally independent **J**-hyper-holomorphic functions. A hypercomplex manifold equipped with an atlas of local **J**-hyper-Spencer coordinate systems is called a hypercomplex manifold of Spencer type m .

Having in mind the interconnection between **J**-hyper-holomorphic functions and **J**-almost-holomorphic ones we derive the analogues of the *Lemmas 1, 2* and *3* of the previous paragraphs. Let us remark that in view that f_α are **J**-almost-holomorphic, and φ_α are **J**-almost-antiholomorphic, the corresponding matrix representations of J^* is as follows (according to *Lemma 1*)

$$\begin{bmatrix} iE_m & * & 0 & * \\ 0 & * & 0 & * \\ 0 & * & -iE_m & * \\ 0 & * & 0 & * \end{bmatrix} \times \begin{bmatrix} -iE_m & * & 0 & * \\ 0 & * & 0 & * \\ 0 & * & iE_m & * \\ 0 & * & 0 & * \end{bmatrix}$$

Analogously, **K**-hyper-Spencer coordinates can be introduced with the help of **K**-hyper holomorphic mappings. The *Proposition 2* remains valid for **J**-holomorphic transition functions and **K**-holomorphic transition functions. When the transition transformations are simultaneously **J**- and **K**-holomorphic it follows that they are affine.

Full coordinate systems defined by $m = (1/2)\dim M_{\mathbb{R}}$ functions which are both **J** and **K** hyper-holomorphic lead to quaternionic manifolds.

0.3 Elliptic Equations

0.3.1 Potential structures on almost-complex manifolds

Let (M, \mathbf{J}) be an almost complex manifold. We shall consider the following globally defined on M Pfaffian form: $\omega = J^* du$, where $u = u(p), p \in M$, is a real-valued smooth (at least of class \mathcal{C}^2) function. In the case the 1-form ω is closed, we will say that ω defines a *potential structure* on the almost complex manifold (M, \mathbf{J}) . On each local real coordinate system $(U, x = (x^k)), x^k \in \mathbb{R}, k = 1, \dots, 2n$, we have a matrix representation of \mathbf{J} , i.e. $\mathbf{J} = \| J_j^k(x) \|$, where $J_j^k(x)$ are smooth real functions on U . By J_j is denoted the j -row of the mentioned matrix and ∇u is the gradient of u . It is easy to see

$$J du = \sum_{q=1}^{2n} (J_q \cdot \nabla u) dx^q$$

where

$$J_q \cdot \nabla u = \sum_{p=1}^{2n} J_q^p \frac{\partial u}{\partial x^p},$$

For each potential structure on (M, \mathbf{J}) the following two statements hold.

Consequence 1. On every simply connected domain $\Omega \subset M$ it holds that

$$\int_{\gamma} J^* du = 0$$

for each closed curve γ in Ω .

Consequence 2. The following system

$$\frac{\partial(J_q \bullet \nabla u)}{\partial x^s} = \frac{\partial(J_s \bullet \nabla u)}{\partial x^q},$$

$s, q = 1, \dots, 2n$, is satisfied locally.

0.3.2 Almost pluri-harmonic functions

By (M, \mathbf{J}, ω) is denoted an almost-complex manifold (M, \mathbf{J}) equipped with potential structure ω . Then the 1-form $\omega = J^* du$ is close, and we have $dJ^* du = 0$. In this case we will say that the function u is an almost-pluriharmonic function. The interconnection between almost-pluriharmonic functions and almost-holomorphic ones (with respect to \mathbf{J}) is like this one between pluriharmonic functions and holomorphic ones. This follows directly from the Cauchy-Riemann equations $J^* du = -dv, J^* dv = du$. Clearly the real part u and the imaginary part v of the almost-holomorphic function $f = u + iv$ are almost-pluriharmonic functions.

0.3.3 Elliptic equations on almost-complex manifolds

We denote by $\Delta_{\mathbf{J}}$ the following differential operator of second order (in terms of coordinates)

$$\Delta_{\mathbf{J}} = \sum_{s,p=1}^{2n} A_{sp} \frac{\partial^2}{\partial x^s \partial x^p} + \sum_{p=1}^{2n} B_p \frac{\partial}{\partial x^p}$$

where

$$A_{sp} = \sum_{q=1}^n (J_q^s J_q^p + \delta_q^s \delta_q^p),$$

and

$$B_p = \sum_{s,q=1}^{2n} J_q^s \left(\frac{\partial J_q^p}{\partial x^s} - \frac{\partial J_s^p}{\partial x^q} \right),$$

δ_q^s, δ_q^p are the Kronecker symbols. Setting $A_J = \|A_{sp}\|$, we obtain

$$A_J = JJ^* + E_{2n}$$

where J^* is the transpose of J and E_{2n} is the unity $2n \times 2n$ matrix.

We emphasize here that now we work with real coordinates, but not with complex self-conjugate ones. However this corresponds to the Spencer type 0. In the other extreme case of Spencer type n we have complex-analytic (holomorphic) coordinates. This is the case of complex analytic manifold with the standard almost-complex structure denoted by \mathbf{S}^0 (it is different from \mathbf{S} in the previous paragraph).

$$-\mathbf{S}^0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \dots \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (n \text{ times})$$

As $S^0(S^0)^* = E_{2n}$ we get $A_{S^0} = 2E_{2n}$ and $\Delta_{\mathbf{S}^0} = 2\Delta$, where Δ is the Laplace operator in $2n$ real variables.

Proposition 3: $\Delta_{\mathbf{J}}$ is an elliptic differential operator.

Proof: It is sufficient to consider the following inequality

$$\sum_{s=1}^{2n} \sum_{p=1}^{2n} A_{sp} \xi_s \xi_p = \sum_{q=1}^{2n} \left(\sum_{s=1}^{2n} J_q^s \xi_s \right)^2 + \sum_{q=1}^{2n} \left(\sum_{s=1}^{2n} \delta_q^s \xi_s \right)^2 \geq \sum_{q=1}^{2n} \xi_q^2. \quad \blacksquare$$

Considering the PDE

$$\Delta_{\mathbf{J}} u = 0,$$

we can state the following

Theorem : Each almost pluriharmonic function u satisfies locally the equation $\Delta_{\mathbf{J}} u = 0$

Proof: Let u be almost pluriharmonic, i.e. $dJdu = 0$, or the 1-form J^*du is closed. According to the previous paragraph u satisfies locally the following system of PDEs

$$\frac{\partial(J_q \bullet \nabla u)}{\partial x^s} = \frac{\partial(J_s \bullet \nabla u)}{\partial x^q},$$

$s, q = 1, \dots, 2n$. Now replacing

$$J_q \bullet \nabla u = \sum_{p=1}^{2n} J_q^p \frac{\partial u}{\partial x^p} \quad \text{and} \quad J_s \bullet \nabla u = \sum_{p=1}^{2n} J_s^p \frac{\partial u}{\partial x^p}$$

in (4) we obtain the system

$$\sum_{p=1}^{2n} \left(\frac{\partial(J_q^p \frac{\partial u}{\partial x^p})}{\partial x^s} - \frac{\partial(J_s^p \frac{\partial u}{\partial x^p})}{\partial x^q} \right) = 0,$$

$k, s = 1, \dots, 2n$. Multiplying each of the above written equations by J_q^s and summing with respect to s we obtain

$$\sum_{p=1}^{2n} \sum_{s=1}^{2n} \left(J_k^p J_q^s \frac{\partial^2 u}{\partial x^s \partial x^p} - J_q^s J_s^p \frac{\partial^2 u}{\partial x^k \partial x^p} \right) = \sum_{p=1}^{2n} \sum_{s=1}^{2n} J_q^s \left(\frac{\partial J_s^p}{\partial x^k} - \frac{\partial J_k^p}{\partial x^s} \right) \frac{\partial u}{\partial x^p}.$$

As we have

$$\sum_{s=1}^{2n} J_q^s J_s^p = -\delta_q^p$$

and

$$\frac{\partial^2 u}{\partial x^k \partial x^p} = \sum_{s=1}^{2n} \delta_k^s \frac{\partial^2 u}{\partial x^s \partial x^p},$$

we obtain

$$\sum_{p=1}^{2n} \sum_{s=1}^{2n} (J_k^p J_q^s + \delta_q^p \delta_k^s) \frac{\partial^2 u}{\partial x^s \partial x^p} = \sum_{p=1}^{2n} \sum_{s=1}^{2n} J_q^s \left(\frac{\partial J_s^p}{\partial x^k} - \frac{\partial J_k^p}{\partial x^s} \right) \frac{\partial u}{\partial x^p}.$$

Now taking $q = k$ and summing with respect to k we get exactly

$$\Delta_{\mathbf{J}} u = 0. \quad \blacksquare$$

In the case $\mathbf{J} = \mathbf{S}^0$ the above written equation is just the classical Cauchy-Riemann system.

Consequences:

1. Each almost pluriharmonic function and respectively every almost holomorphic function of class \mathcal{C}^2 on a smooth manifold are of class \mathcal{C}^∞ too.

2. For connected smooth manifolds the maximum principle holds.
 3. In the case of real analytic manifold M , equipped with real-analytic structure \mathbf{J} , each \mathbf{J} -pluriharmonic and each \mathbf{J} -almost-holomorphic function is real analytic.
 4. In the case of connected real analytic manifold M with real-analytic structure \mathbf{J} the principle of unicity of the analytic continuation holds.
- Remark:* This theorem is inspired from the paper [BKW]. The first announcement is in [DM]

0.3.4 The equation $dJ^*du = 0$ in terms of vector fields - commutators and anti-commutators

Applying the well known formula

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]), \omega \text{ is 1-form, } X, Y \text{ are vector fields}$$

to the 1-form $\omega = \mathbf{J}du$ we present the equation (2) in terms of expressions of vector fields, namely

$$[X, Y]_{\mathbf{J}}(u) = \mathbf{J}[X, Y](u)$$

where $[X, Y]_{\mathbf{J}} \stackrel{def}{=} X \circ \mathbf{J}Y - Y \circ \mathbf{J}X$. It is to remark that $[X, Y]_{\mathbf{J}}$ is not a vector field. For instance:

$$[X, Y]_{\mathbf{J}}(fh) = [X, Y]_{\mathbf{J}}(f)h + f[X, Y]_{\mathbf{J}}(h) + X(f)(\mathbf{J}Y)(h) - (\mathbf{J}X)(f)Y(h) + X(h)(\mathbf{J}Y)(f) - (\mathbf{J}X)(h)Y(f)$$

Some properties of $[X, Y]_{\mathbf{J}}$

Considering the natural splitting

$$\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M$$

we can take the restriction of $[X, Y]_{\mathbf{J}}$ on $T^{1,0}M$. This means that

$$\mathbf{J}X = iX \quad \text{and} \quad \mathbf{J}Y = iY$$

where $X, Y \in T^{1,0}M$. So we have

$$[X, Y]_{\mathbf{J}} = X \circ (iY) - Y \circ (iX) = i[X, Y]$$

Analogously

$$[X, Y]_{\mathbf{J}} = (-i)[X, Y] \quad \text{on } T^{0,1}M$$

Now we take $X \in T^{1,0}M$ and $Y \in T^{0,1}M$

$$[X, Y]_{\mathbf{J}} = X \circ (iY) - Y \circ (-iX) = i(X \circ Y + Y \circ X) = i\{X, Y\}$$

Here $\{X, Y\}$ denotes the anticommutator of X and Y . Analogously, if $X \in T^{0,1}M$ and $Y \in T^{1,0}M$:

$$[X, Y]_{\mathbf{J}} = -i\{X, Y\}$$

0.3.5 Potential structures on hypercomplex manifolds

On a hypercomplex manifold $(M, \mathbf{J}, \mathbf{K})$ we can consider two separate potential structures, namely

$$\omega_1 = \mathbf{J}^* du \quad \text{and} \quad \omega_2 = \mathbf{K}^* d\zeta$$

or the sum

$$\omega = \mathbf{J}^* du + \mathbf{K}^* d\zeta$$

The corresponding almost-pluriharmonic functions u, v, ζ, η satisfy the equations:

$$d\mathbf{J}^* du = d\mathbf{J}^* dv = 0 \quad \text{and} \quad d\mathbf{J}^* d\zeta = d\mathbf{J}^* d\eta = 0$$

We have also the natural defined elliptic operators $\Delta_{\mathbf{J}}$ and $\Delta_{\mathbf{K}}$. According to the proved theorem:

$$d\mathbf{J}^* du = d\mathbf{J}^* dv = 0 \quad \Rightarrow \quad \Delta_{\mathbf{J}} u = \Delta_{\mathbf{J}} v = 0$$

and

$$d\mathbf{K}^* d\zeta = d\mathbf{K}^* d\eta = 0 \quad \Rightarrow \quad \Delta_{\mathbf{K}} \zeta = \Delta_{\mathbf{K}} \eta = 0$$

For the sum $\omega = \mathbf{J}^* du + \mathbf{K}^* d\zeta$ a pair of functions (u, ζ) appears, namely the solutions of the following second order equation:

$$d\mathbf{J}^* du + d\mathbf{K}^* d\zeta = 0$$

In terms of vector fields the above written equations seem as follows

$$[X, Y]_{\mathbf{J}} u = \mathbf{J}[X, Y](u) \quad \text{and} \quad [X, Y]_{\mathbf{K}} u = \mathbf{K}[X, Y](u)$$

0.4 Generation of almost-complex structures

0.4.1 Remarks on the local equation of almost-holomorphic functions

Let (M, \mathbf{J}) be an almost-complex manifold, $\dim M = 2n$. Having in mind the question of the local integration of the equation $\mathbf{J}^* df = idf$, we shall examine how "far away" a non-integrable almost complex structure \mathbf{J} is from the classical complex structure related with the standard almost-complex structure \mathbf{S} .

Let p be a point of M . Taking an open neighborhood U of the point p , small enough, we can accept that U is a neighborhood of the origin in \mathbb{R}^{2n} (p to be the origin). Now we shall replace \mathbf{J} by its matrix representation J on U and J^* will denote the transposed matrix. We will use general real coordinates $x = (x^1, \dots, x^{2n}) \in \mathbb{R}^{2n}$. Let G denotes a non-degenerate $(2n \times 2n)$ matrix, such that $G^{-1} J^*(0) G = S^*$, where S^* is the transposed matrix of S ,

$$S = \begin{bmatrix} 0 & -E_n \\ E_n & 0 \end{bmatrix}, E_n \text{ being the unit } n \times n \text{ matrix.}$$

For $x \in U$ we set:

$$G^{-1}J(x)G = \begin{bmatrix} A(x) & B(x) + E_n \\ C(x) - E_n & D(x) \end{bmatrix}$$

$A(x), B(x), C(x), D(x)$ are $n \times n$ matrices.

Clearly we have:

$$\begin{bmatrix} A(x) & B(x) + E_n \\ C(x) - E_n & D(x) \end{bmatrix} = S^* \text{ and } A(0) = B(0) = C(0) = D(0) = 0$$

Moreover, we have $(G^{-1}J(x)G)^2 = -E_{2n}$, which implies the following identities:

$$\begin{aligned} A^2(x) + (B(x) + E_n)(C(x) - E_n) &= -E_n \\ A(x)(B(x) + E_n) + (B(x) + E_n)D(x) &= 0_n \\ (C(x) - E_n)A(x) + D(x)(C(x) - E_n) &= 0_n \\ (C(x) - E_n)(B(x) + E_n) + D^2(x) &= -E_{2n} \end{aligned}$$

From the last system it follows that locally is valid:

$$\begin{aligned} A(x) &= -(C(x) - E_n)^{-1}D(x)(C(x) - E_n) \\ B(x) + E_n &= -(C(x) - E_n)^{-1}(D^2(x) + E_n) \end{aligned}$$

Indeed, as

$$\det(C(0) - E_n) = (-1)^n \neq 0$$

the inverse matrix $(C(x) - E_n)^{-1}$ exists in some neighborhood of the origin $\mathbf{0} \in \mathbb{R}^n$.

Now let's consider the equation $(J^* - iE_{2n})df = 0$. It follows that

$$(G^{-1}J^*G - iE_{2n})df = 0$$

and also

$$\begin{bmatrix} A(x) - iE_n & B(x) + E_n \\ C(x) - E_n & D(x) - iE_n \end{bmatrix} df = 0$$

Proposition: The following block matrix identity is valid:

$$\begin{bmatrix} A(x) - iE_n & B(x) + E_n \end{bmatrix} = (A(x) - iE_n)(C(x) - E_n)^{-1} \begin{bmatrix} C(x) - E_n & D(x) - iE_n \end{bmatrix}$$

Proof: Let consider the right side of the identity:

$$\begin{aligned} &(A(x) - iE_n)(C(x) - E_n)^{-1} \begin{bmatrix} C(x) - E_n & D(x) - iE_n \end{bmatrix} = \\ &= \begin{bmatrix} A(x) - iE_n & (A(x) - iE_n)(C(x) - E_n)^{-1}(D(x) - iE_n) \end{bmatrix} \end{aligned}$$

But:

$$\begin{aligned} &(A(x) - iE_n)(C(x) - E_n)^{-1}(D(x) - iE_n) = B(x) + E_n, \\ &\text{as } A(x) = -(C(x) - E_n)^{-1}D(x)(C(x) - E_n). \end{aligned}$$

The last equality becomes:

$$\begin{aligned} &(- (C(x) - E_n)^{-1}D(x)(C(x) - E_n) - iE_n)(C(x) - E_n)^{-1}(D(x) - iE_n) = \\ &= (C(x) - E_n)^{-1}(-D(x) - iE_n)(C(x) - E_n)(C(x) - E_n)^{-1}(D(x) - iE_n) = \\ &= -(C(x) - E_n)^{-1}(D(x) + iE_n)(D(x) - iE_n) = \end{aligned}$$

$$= -(C(x) - E_n)^{-1}(D^2(x) + iE_n) = B(x) + E_n. \blacksquare$$

Corollary: The first n equations of the considered system

$$(J^* - iE_{2n})df = 0$$

follow from the last n ones. So we obtain that locally this system is equivalent to the next one:

$$\begin{bmatrix} C(x) - E_n & D(x) - iE_n \end{bmatrix} df = 0$$

or:

$$\begin{bmatrix} E_n & (C(x) - E_n)^{-1}(D(x) - iE_n) \end{bmatrix} df = 0$$

Setting $P(x) \stackrel{def}{=} (C(x) - E_n)^{-1}D(x)$ and $Q(x) \stackrel{def}{=} (C(x) - E_n)^{-1}$, we receive the following block matrix form of the considered equation of almost holomorphic functions:

$$\begin{bmatrix} E_n & P(x) + iQ(x) \end{bmatrix} df = 0.$$

0.4.2 Local reconstruction of J by the matrices P and Q

We will use the following equalities:

$$\begin{aligned} C - E_n &= Q^{-1}; \quad D = Q^{-1}P; \quad A = -QQ^{-1}PQ = -PQ^{-1}; \\ B + E_n &= -Q((Q^{-1}P)^2 + E_n) = -PQ^{-1}P - E_n. \end{aligned}$$

The matrix J can be reconstructed as follows:

$$J = \begin{bmatrix} -PQ^{-1} & -PQ^{-1}P - Q \\ Q^{-1} & Q^{-1}P \end{bmatrix} \quad (*)$$

The mentioned reconstruction (*) can be considered as a generation of the matrix representation of J on the open set U by the pair of matrices (P, Q) . Denoting by $\mathcal{M}(U, n)$ the algebra of all $(n \times n)$ -matrices equipped with the topology of coordinate convergence, we can consider the Cartesian product $\mathcal{M}(U, n) \times \mathcal{M}(U, n)$ with the product topology as a continuous family which generates the set $\mathcal{J}(U, 2n)$ of all $(2n \times 2n)$ -matrices J , which verify the matrix equation

$$J^2 + E_{2n} = 0,$$

as a kind of moduli space (locally). More precisely, the following proposition holds

Proposition 4: For each $J \in \mathcal{J}(U, 2n)$ there is a pair $(P, Q) \in \mathcal{M}(U, n) \times \mathcal{M}(U, n)$ such that J is generated by (P, Q) in the sense of the rule (*). Conversely, each pair (P, Q) defines a J according to the rule (*). Each sequence (P_n, Q_n) of elements of $\mathcal{M}(U, n) \times \mathcal{M}(U, n)$ determines a sequence of elements of $\mathcal{J}(U, 2n)$, and the limit of the second sequence corresponds by the rule (*) to the limit of the first sequence.

The proof is clear.

0.4.3 Global reconstruction of J .

The problem of global reconstruction of almost complex structures on a smooth manifold by an appropriate algebraic objects is much more difficult. It seems that an approach can be developed on real-analytic almost complex manifold (M, J) having local matrix representation for J with real-analytic coefficients. Now we shall consider the sheaf of germs of almost complex structures, denoted by $\mathcal{J}(M)$, and the sheaf of germs of pairs of matrices (P, Q) . Supposing that each J can be considered as a global section of the sheaf $\mathcal{J}(M)$, we can develop the rule (*) for germs of $\mathcal{J}(M)$ and germs of pairs (P, Q) at each point $p \in M$. The set of global sections of $\mathcal{J}(M)$ must be generated by the sections of the sheaf of germs of pairs (P, Q) .

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